Fourier transform and distributions

3.1 Ground work, a confession

We have a confession to make.

Recall your first days of learning differential and integral calculus. After learning the basic tools and methods, the first functions you applied the tools of differentiation and integration were a constant, trigonometric functions, etc. We did not follow the same path for Fourier transform. The first examples we dealt with were unusual functions like the rect() function and the triangle function. The reason for this is that we cannot evaluate the Fourier transform of simple functions like 1, sin t, or cos t — yet. With all these functions, we can perform the integrations involved with Fourier transform, but we would be unable to substitute the limits. There are issues involving integrability of these functions. We also have a problem with the very first problem that we have handled. We showed $\mathcal{F}\Pi(w) = \mathrm{sinc}(w/2)$. This gives us the inverse transform $\mathcal{F}^{-1}\mathrm{sinc}(t) = \Pi(t)$. We have also shown, using duality, that $\mathcal{F}\mathrm{sinc}(w) = \Pi(t)$. If we pause a little bit and look deeply at both of these integrals, we would realize that none of these integrals would actually evaluate. The reason is that the sinc() function is a uniformly continuous function, and the rect() function is a piecewise continuous function. Remembering that integration is a summation process, it is mathematically not possible for a uniformly continuous function to integrate and result in a piecewise continuous function. Therefore, there are serious issues involving convergence of these integrals.

These are serious issues which require fundamental change in viewpoints on how we look at integrations in general, and Fourier transform in particular. The issues involved here required serious mathematical deliberations and reinterpretations. To do this, we follow a different line of development that was taking place simultaneously. This is the development of generalized functions, or distributions – typified by the delta function.

3.2 Delta function

Delta function was introduced by British physicist Paul Dirac. Therefore, the function is also, sometimes, referred to as Dirac's delta function; or simply Dirac's delta. The function is defined as

$$\delta(x) = 0 \quad x \neq 0 \tag{3.1}$$

$$\int_{-\infty}^{\infty} \delta(x)dx = 1 \tag{3.2}$$

$$\int_{-\infty}^{\infty} \delta(x)\varphi(x)dx = \varphi(0) \tag{3.3}$$

Strictly speaking, delta function is not a function in mathematical sense, because no function can satisfy the defined conditions. Because it is only defined that provided $x \neq 0$, the value of the function is 0. The function is not defined at $x = 0$. Nevertheless, $\int_{-\infty}^{\infty} \delta(x)dx = 1$. All we can say is that there is a 'spike' at $x = 0$. The 'value' of the 'spike' is not known. Which makes it a little 'dubious' to define. Because the only 'non-zero' value of the function is at $x = 0$, the function is also called an *impulse function*. Despite its dubious nature, the function has proven to be priceless in dealing with systems where there are forcing functions which are impulsive in nature. Examples include a point charge in space, an electronic circuit excited suddenly, digital circuits, sampling an analog signal, etc.

The matter was contradictory for mathematicians, scientists and engineers together. On one hand, delta function was very successful in handling problems involving impulsive input. On the other hand, the mathematical foundation of the delta function was very 'shaky'. To build the delta function on a more solid ground, the function was redefined as a limiting case of other 'nice' functions. To give an example here, we would choose a modified form of the rect() function defined as follows.

$$\begin{aligned} \frac{1}{\varepsilon}\Pi_\varepsilon(x) &= 1 \qquad -\frac{\varepsilon}{2} < x < \frac{\varepsilon}{2} \\ &= 0 \qquad \textit{elsewhere} \end{aligned}$$

Performing the integration of this function

$$\frac{1}{\varepsilon}\int_{-\infty}^{\infty}\Pi_{\varepsilon}(x)dx = \frac{1}{\varepsilon}\int_{-\varepsilon/2}^{\varepsilon/2}1dx = 1 \tag{3.4}$$

This retrieves the property stated in (3.2). To retrieve the last property,

$$\frac{1}{\varepsilon}\int_{-\infty}^{\infty}\Pi_{\varepsilon}(x)\varphi(x)dx = \frac{1}{\varepsilon}\int_{-\varepsilon/2}^{\varepsilon/2}\varphi(x)dx$$

To perform the integration, we expand $\varphi(x)$ in Taylor series around $x_0 = 0$.

$$\begin{aligned}\frac{1}{\varepsilon}\int_{-\infty}^{\infty}&\Pi_{\varepsilon}(x)\varphi(x)dx\\ &= \frac{1}{\varepsilon}\int_{-\varepsilon/2}^{\varepsilon/2}\left(\varphi(0)+\varphi'(0)x+\varphi''(0)\frac{x^2}{2!}+\cdots\right)dx\\ &= \frac{1}{\varepsilon}[(\varphi(0)+O(\varepsilon^3))]_{-\varepsilon/2}^{\varepsilon/2} = \varphi(0)+O(\varepsilon^2)\end{aligned}$$

Now, if we take the limit $\varepsilon \to 0$ we get

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{-\infty}^{\infty}\Pi_{\varepsilon}(x)\varphi(x)dx = \varphi(0) \tag{3.5}$$

This retrieves the property stated in (3.3). We observe that we have been able to retrieve the important properties of the delta function without any dubious definition appearing in (3.1)-(3.3). Operationally, equation (3.3) was fine, because in practical applications, the delta function hardly ever appeared by itself. Almost invariably, it appears being paired with another function like $\varphi(x)$ as appearing in (3.3). Equations (3.3) and (3.5) are telling us the same story that 'operationally the delta function retrieves the value of the function $\varphi(x)$ at 0'. From a mathematical point of view though (3.3) could not make sense because of the dubious definition of $\delta(x)$, but there is no ambiguity with equation (3.5).

We also observe another very important side-effect of the development of (3.5). We notice that though $\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Pi_{\varepsilon}(x)$ itself does not make any sense mathematically, but when $\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Pi_{\varepsilon}(x)$is 'paired' with another function $\varphi(x)$, it is unambiguous, and makes perfect sense.

This little exercise solved the problem of the ambiguous nature of the delta function. It also gave a clue as to how to handle functions in Fourier transform which we would not be able to do otherwise. This involves computing the Fourier transform, the inverse Fourier transform, derivatives, etc.

Let us try to state what our problem is, and how we intend to address the issue. Fourier transform is a very rich subject which is based on very solid theoretical foundation. It has proven priceless in signal analysis. Even with signals for which we could not obtain the Fourier transform, or the inverse Fourier transform, it was possible to obtain the basic essence of the signal using approximations of the signals. Therefore, it was felt that the problem was not with the way Fourier transform, and its inverse was defined, but rather the problem was with the way these were evaluated. What became necessary was to take a step back and rework much of the groundwork that led to evaluation of the Fourier transform and its inverse. Let us try to emphasize what is going to happen here. This is perhaps the next biggest theory of mathematics you are learning after calculus. In essence, we would redefine much of the groundwork of differential and integral calculus. So, buckle up, and let us proceed.

3.3 The way to proceed, the theory

We want our theory to be able to provide us with the following abilities.

- to handle signals like the delta function, sine, cosine, step function and other functions which are necessary for engineering analysis;
- compute Fourier transform and inverse Fourier transform for these signals;
- Fourier inversion can be applied.

The objectives would be achieved in the following way.

1. We single out a collection of functions $\mathscr{S}$ for which convergence of the Fourier integrals is assured, for which the function *and* its

Fourier transform are both in $\mathcal{S}$, and for which Fourier inversion works. Furthermore, Parseval's identity holds.
2. $\mathcal{S}$ forms a class of *test functions* which, in turn, serve to define a larger class of *generalized functions* or *distributions*, called, for this class of test functions the *tempered distributions*, $\mathcal{T}$. Precisely because $\mathcal{S}$ was chosen to be the ideal Fourier friendly space of classical signals, the tempered distributions are likewise well suited for Fourier methods.
3. The Fourier transform and its inverse will be defined so as to operate on these tempered distributions, and they operate to produce distributions of the same type. Thus, the inverse Fourier transform can be applied, and the Fourier inversion theorem holds in this setting.

A function $f(x)$ can have the above properties if it has two following properties.

1. $f(x)$ is infinitely differentiable
2. For all positive integers m and n, $\left|x^m \frac{d^n}{dx^n} f(x)\right| \to 0$ as $x \to \pm\infty$.

In words, any positive power of x decreases more rapidly than any derivative of f.

Let us try to analyze why the above two conditions are such good ideas.

1. If $f(x) \in \mathcal{S}$, $\mathcal{F}f(w) \in \mathcal{S}$. In words, if $f(x)$ is rapidly decreasing, Fourier transform of $f(x)$ would also be rapidly decreasing.
2. Fourier inversion works in $\mathcal{S}$.
3. Parseval's identity holds in $\mathcal{S}$.

We will prove the last propositions. Earlier, we saw Parseval's identity for Fourier series. For Fourier transform, Parseval's identity states

$$\int_{-\infty}^{\infty} |\mathcal{F}f(w)|^2 dw = \int_{-\infty}^{\infty} |f(t)|^2 dt \tag{3.6}$$

We would provide a more general version

$$\int_{-\infty}^{\infty} \mathcal{F}f(w)\overline{\mathcal{F}g(w)} dw = \int_{-\infty}^{\infty} f(t)\overline{g(t)} dt \tag{3.7}$$

Where $\overline{\mathcal{F}g(w)}$ is the complex conjugate of $\mathcal{F}g(w)$ and $\overline{g(t)}$ is the complex conjugate of $g(t)$.

We will write $g(t)$ as inverse Fourier transform of $\mathcal{F}g(w)$

$$g(t) = \int_{-\infty}^{\infty} \mathcal{F}g(w)e^{iwt}dw$$

Taking the complex conjugate

$$\overline{g(t)} = \int_{-\infty}^{\infty} \overline{\mathcal{F}g(w)}e^{-iwt}dw \tag{3.8}$$

We can do this because, if f and g are in $\mathcal{S}$, both Fourier transform, and inverse Fourier transform would compute. Substituting (3.8) in (3.7), we obtain

$$\int_{-\infty}^{\infty} \mathcal{F}f(w)\overline{\mathcal{F}g(w)}dw = \int_{-\infty}^{\infty} f(t)\left(\int_{-\infty}^{\infty} \overline{\mathcal{F}g(w)}e^{-iwt}dw\right)dt$$

Now changing the order of integration, we get

$$\int_{-\infty}^{\infty} \mathcal{F}f(w)\overline{\mathcal{F}g(w)}dw = \int_{-\infty}^{\infty} \overline{\mathcal{F}g(w)}\left(\int_{-\infty}^{\infty} f(t)e^{-iw}\ dt\right)dw$$

We recognize the integral within the bracket as $\mathcal{F}f(w)$. Doing so, we get the left-hand side. A special case of this integral, if $g = f$, we obtain the Parseval's identity.

Parseval's identity is also known as *Plancheral's identity*, or *Rayleigh identity*. For engineering applications, Parseval's identity has a very powerful physical significance. Equation (3.6) tells us that the total energy of a signal is same in the time domain and in the frequency domain.

The class of function in $\mathcal{S}$ is often referred to as *Schwartz function*. Examples of Schwartz function include the Gaussian, and functions with *compact support*.

3.4. Distributions defined

We have completed the foundation on which we would define a generalized function, or a distribution. To define a generalized function, we take our clue from the way the problem involving the delta function was solved. Recalling that the way the delta function was defined in (3.1)-(3.3), there were lots of ambiguities. But later we defined the delta function as a limiting case of the rect() function defined between –ε/2 and ε/2 in the limit as $\varepsilon \to 0$. We found out that though $\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Pi_\varepsilon(x)$ itself did not make any sense mathematically, but when $\frac{1}{\varepsilon}\Pi_\varepsilon(x)$ was 'paired' with another function $\varphi(x)$, $\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{-\infty}^{\infty}\Pi_\varepsilon(x)\varphi(x)dx$ was unambiguous, and made perfect sense. Therefore, to define a distribution, we start with a class of 'test functions' $\varphi(x)$. These functions would be of the category of Schwartz function, i.e. infinitely differentiable, and rapidly decreasing. These functions would be chosen based on the area of application. Associated with the test function would be a class of tempered distributions, $f(x)$. To define a distribution, we will 'pair' $f(x)$ with $\varphi(x)$. The function $\varphi(x)$ will be referred to as the 'test function'. In terms of notation, the pairing will be indicated by $\langle f(x), \varphi(x)\rangle$, and is defined mathematically by integration.

$$\langle f(x), \varphi(x)\rangle = \int_{-\infty}^{\infty} f(x)\varphi(x)dx \tag{3.9}$$

Let us try to ascertain what we are trying to achieve. The function $f(x)$ is such that the integral $\int_{-\infty}^{\infty} f(x)dx$ is ambiguous, but when paired with $\varphi(x)$, the integral is unambiguous. For example, $\int_{-\infty}^{\infty} 1\, dx$ would not compute, but $\int_{-\infty}^{\infty} 1\, \varphi(x)dx$ has no problem. It is worth mentioning why it was enforced again and again that the test function $\varphi(x)$ is infinitely differentiable and rapidly decreasing. It is ambiguous to substitute the limits $-\infty$ and ∞ in the integral $\int_{-\infty}^{\infty} 1\, dx$; but we would not have any such problem in $\int_{-\infty}^{\infty} 1\, \varphi(x)dx$, as because of the rapidly reducing nature of $\varphi(x)$, we are sure that $\varphi(x) \to 0$, as $x \to \pm\infty$.

Let us pause for a moment and try to understand what we are doing. There is a problem with the convergence of the integral $\int_{-\infty}^{\infty} f(x)dx$ in the traditional sense of integral calculus. If it was possible for us to compute the integral with the limits, the result would have been a number. The number would have different physical significance depending upon what the function $f(x)$ signifies. It could have been length, area, volume, work done, etc. The integral $\int_{-\infty}^{\infty} f(x)\varphi(x)dx$ would also result in a number. The number would have different physical significance, depending upon the physical significance of $f(x)$ and how we have defined $\varphi(x)$. Equation (3.9) gives a new definition of the integration of the function $f(x)$ in situations where the traditional definition fails. To put this in greater perspective, let us retrieve the definition of the delta function using distributions. Earlier, we saw (with a clean algebra)

$$\int_{-\infty}^{\infty} \delta(x)\varphi(x)dx = \varphi(0)$$

In terms of our definition of a distribution, this can be written as

$$\langle \delta(x), \varphi(x) \rangle = \varphi(0) \tag{3.10}$$

Going back to the explanation that we were just proposing, equation (3.10) states that the delta function operating on a signal $\varphi(x)$ returns the value of the signal at time 0. As a corollary to this, we can also evaluate

$$\langle \delta_a(x), \varphi(x) \rangle = \varphi(a) \tag{3.11}$$

The term δ_a is the shifted delta function $\delta(x - a)$, delta function defined at $x = a$. Shifted delta function operating on a function $\varphi(x)$ returns the value of the function at $x = a$. Invaluable tool for sampling.

3.5 Fourier transform

We have described the basic theory that is needed for our purpose. We are ready to learn how to evaluate the Fourier transform of a distribution. It would be amazing how effortless our work would be from this point.

Once again, we have a tempered distribution $f(x)$, whose Fourier transform we intend to compute. We understand that we would be unable to evaluate the Fourier transform $\mathcal{F}f$. So, we pair $\mathcal{F}f(x)$ with a test function $\varphi(x)$. We would also ensure that $\varphi(x)$ is a Schwartz function. So, we can not only take the Fourier transform of $\varphi(x)$, $\mathcal{F}\varphi$ would also be a Schwartz function, meaning that $\mathcal{F}\varphi$ is also infinitely differentiable and rapidly reducing. So,

$$\langle \mathcal{F}f, \varphi \rangle = \int_{-\infty}^{\infty} \mathcal{F}f(x)\varphi(x)dx$$

We expand $\mathcal{F}f(x)$ using the definition of Fourier transform,

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x)e^{-iwx}dx \right) \varphi(x)dx$$

We remove the bracket, regroup the variables

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \varphi(x)e^{-iw}\ \ dx \right) f(x)dx = \int_{-\infty}^{\infty} f(x)\mathcal{F}\varphi(x)dx$$

$$= \langle f, \mathcal{F}\varphi \rangle$$

To summarize

$$\langle \mathcal{F}f, \varphi \rangle = \langle f, \mathcal{F}\varphi \rangle \tag{3.12}$$

In words, Fourier transform of a tempered distribution $\mathcal{F}f(x)$ paired with a test function $\varphi(x)$ is equal to the tempered distribution $f(x)$ paired with the Fourier transform of the test function $\mathcal{F}\varphi(x)$.

Why will this compute? We repeat once again that we have a problem computing the Fourier transform of $f(x)$. Therefore, we 'paired' it with $\varphi(x)$. Wo do not have any problem computing the Fourier transform of $\varphi(x)$ because we chose it as 'Fourier friendly'.

Similarly, we can define the inverse Fourier transform

$$\langle \mathcal{F}^{-1}f, \varphi \rangle = \langle f, \mathcal{F}^{-1}\varphi \rangle \tag{3.13}$$

Fourier inversion would also work

$$\langle \mathcal{F}\mathcal{F}^{-1}f, \varphi \rangle = \langle f, \mathcal{F}\mathcal{F}^{-1}\varphi \rangle \tag{3.14}$$

Enough of theory, let us evaluate some Fourier transforms.

Example
Fourier transform of the delta function, δ.
$\langle \mathcal{F}\delta, \varphi\rangle = \langle \delta, \mathcal{F}\varphi\rangle$

From the definition of the delta function (3.10),

$$\langle \delta(x), \varphi(x)\rangle = \varphi(0) \tag{3.10}$$

Therefore

$$\langle \delta, \mathcal{F}\varphi\rangle = \mathcal{F}\varphi(0) = \int_{-\infty}^{\infty} \varphi(x)e^{-i0x}dx = \int_{-\infty}^{\infty} 1\cdot\varphi(x)dx$$
$$= \langle 1, \varphi\rangle$$

Therefore, we have
$\langle \mathcal{F}\delta, \varphi\rangle = \langle 1, \varphi\rangle$
Therefore, we obtain $\mathcal{F}\delta = 1$.

Example
$f(x) = e^{iax}$

$$\langle \mathcal{F}e^{iax}, \varphi\rangle = \langle e^{iax}, \mathcal{F}\varphi\rangle = \int_{-\infty}^{\infty} e^{iax}\mathcal{F}\varphi(x)dx$$

Pay close attention to the last integral. It is inverse Fourier Transform of Fourier transform of $\varphi(x)$ evaluated at a. This is $\varphi(a)$. Therefore
$\langle \mathcal{F}e^{iax}, \varphi\rangle = \varphi(a) = \langle \delta_a, \varphi\rangle$
Therefore, $\mathcal{F}e^{iax} = \delta_a$.

Special case for $a = 0$. $\mathcal{F}(1) = \delta$.

We have computed a remarkable pair

$$\mathcal{F}\delta = 1 \tag{3.15}$$
$$\mathcal{F}(1) = \delta \tag{3.16}$$

These two simple statements have remarkable and far-reaching implications in science and engineering. We mention a few

- In quantum mechanics, position of a particle and its momentum are Fourier transforms of each other. Therefore, according (3.15), if we fix the position of the particle, its momentum would be spread, and the reverse. We just expressed the Heisenberg uncertainty principle.
- A very thin slit acts as Fourier transform for monochromatic light. Therefore, if we pass a monochromatic (coherent) light through a very thin (of-the-order of wavelength of a photon) slit, we will observe a very bright pattern on the screen. A slit wider than that of the order of wavelength of a photon would produce dark-and-light diffraction pattern on the screen. Remember the rect() function and its Fourier transform sinc() – Figures (2.3) and (2.4) from Lecture 2?

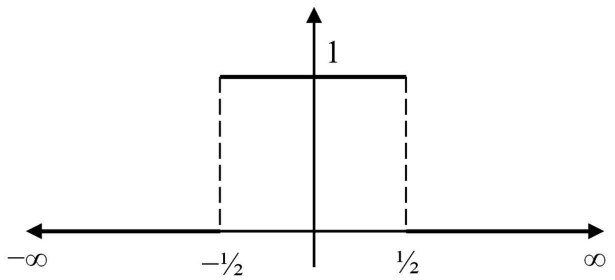

Figure 2.3 The rect() function $\Pi(x)$.

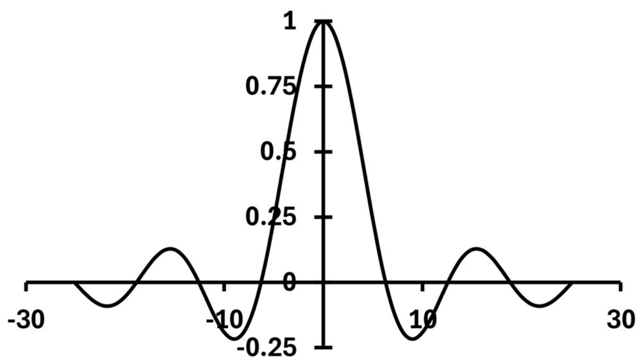

Figure 2.4 Fourier transform of rect().

Example

$f(x) = \cos ax$

$$\mathcal{F}(\cos ax) = \frac{1}{2}\mathcal{F}\left(e^{iax} + e^{-iax}\right) = \frac{1}{2}(\delta_a + \delta_{-a})$$

Similarly,

$$\mathcal{F}(\sin ax) = \frac{1}{2i}\mathcal{F}\left(e^{iax} - e^{-iax}\right) = \frac{1}{2i}(\delta_a - \delta_{-a})$$

3.6 Derivatives of distributions

How do we define the derivative of a tempered distribution? We use the same pairing with a test function $\varphi(x)$ and define the integral.

$$\langle f', \varphi(x)\rangle = \int_{-\infty}^{\infty} f'(x)\varphi(x)dx$$

Using integration by parts

$$= [f(x)\varphi(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\varphi'(x)dx$$

Because of the properties set for a test function, $\varphi(x) \to 0$ as $x \to \pm\infty$. The first part of the result becomes 0. Therefore,

$$= -\int_{-\infty}^{\infty} f(x)\varphi'(x)dx = -\langle f(x), \varphi'(x)\rangle$$

Therefore, the derivative of a tempered distribution exists as another distribution.

$$\langle f', \varphi(x)\rangle = -\langle f(x), \varphi'(x)\rangle \qquad (3.17)$$

Example

Derivative of the Heaviside unit step function defined as

$$\begin{aligned} u(x) &= 1 \quad x > 0 \\ &= 0 \quad x \leq 0 \end{aligned}$$

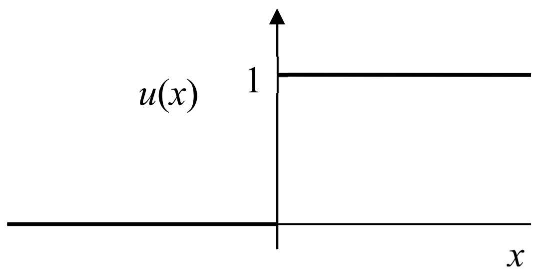

Figure 3.1 Heaviside unit step function

The function is shown graphically in figure 3.1 Therefore, using (3.17), the first derivative,

$$\langle u'(x), \varphi(x)\rangle = -\langle u(x), \varphi'(x)\rangle = -\int_{-\infty}^{\infty} u(x)\varphi'(x)dx$$

$$= -\int_{0}^{\infty} \varphi'(x)dx = -[\varphi(x)]_{0}^{\infty} = \varphi(0) = \langle \delta, \varphi(x)\rangle$$

Therefore $u'(x) = \delta$.

Shifted unit function $u_a(x)$ is defined as $u(x - a)$. From the above result, we also get $u'(x - a) = \delta_a$.

Example

We compute the derivative of the signum function which is defined as

$$\begin{aligned} \mathrm{sgn}(x) &= 1 \quad x > 0 \\ &= 0 \quad x = 0 \end{aligned}$$

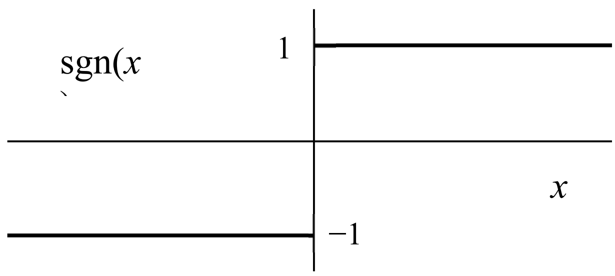

Figure 3.2 The signum function, sgn(x).

$= -1 \qquad x > 0$

The graph of sgn(x) is shown in figure 3.2. To evaluate its first derivative, we evaluate

$$\begin{aligned}\langle sgn'(x), \varphi(x)\rangle &= -\langle sgn(x), \varphi'(x)\rangle \\ &= -\int_{-\infty}^{\infty} sgn(x)\varphi'(x)dx \\ &= -\left[\int_{-\infty}^{0}(-1)\varphi(x)dx + \int_{0}^{\infty}(1)\varphi(x)dx\right] \\ &= [\varphi(x)]_{-\infty}^{0} - [\varphi(x)]_{0}^{\infty} = 2\varphi(0) = \langle 2\delta, \varphi\rangle\end{aligned}$$

Therefore, $\text{sgn}'(x) = 2\delta$.

Shifted signum function $\text{sgn}_a(x)$ is defined as $\text{sgn}(x - a)$.

To appreciate the power of distributions over functions, we must understand the following two points.

(a) Both the distributions we studied above, are 'not differentiable' in the sense of a function in differential calculus. In fact, both the functions are discontinuous. Yet we had no difficulty in finding the first derivative when we defined the same functions as distributions.

(b) It is very important to observe that the normal principle of 'any function is the integration of its derivative' would not necessarily apply to distributions. We observe that the first derivative of the unit step function is the delta function; but the integration of the delta function is not the unit step function.

We can apply the results of derivatives of distributions to evaluate Fourier transform of the Heaviside function and the signum function. These results depend upon the application of the derivative theorem we derived in section 2.4.5.

$$\mathcal{F}f'(w) = iw\,\mathcal{F}f(w)$$

$$(\mathcal{F}f)'(w) = \mathcal{F}[-ixf(x)](w)$$

Example

Fourier transform of sgn(x).

We have evaluated that

$$\text{sgn}'(x) = 2\delta$$

Taking the Fourier transform on both sides we obtain

$$\mathcal{F}[\text{sgn}'(x)] = \mathcal{F}(2\delta) = 2$$

Also from the first of the derivative formula, we obtain

$$\mathcal{F}[\text{sgn}'(x)] = iw\,\mathcal{F}[\text{sgn}(x)]$$

Therefore

$$iw\,\mathcal{F}[\text{sgn}(x)] = 2$$

or $$\mathcal{F}[sgn(x)] = \frac{2}{iw}$$

Example

Fourier transform of the Heaviside unit step function $u(x)$.

We have defined $u(x)$ as

$$u(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

For our convenience, the function can also be defined as

$$u(x) = \tfrac{1}{2}[1 + \text{sgn}(x)]$$

Taking Fourier transform on both sides, we obtain

$$\mathcal{F}[u(x)] = \frac{1}{2}\left(\mathcal{F}(1) + \mathcal{F}(sgn(x))\right) = \frac{1}{2}\left(\delta + \frac{2}{iw}\right)$$

Why are we doing all this? Next we are going to investigate applications of Fourier transform.